Numerical simulation of homogeneous and isotropic universe given by Friedmann equations

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Aim of the project

This project aims to provide a mathematical framework for modeling the Universe. By analyzing cosmological data we compute a set of parameters that, together with the mathematical model, describe the past and future evolution of the Universe.

We will consider the universe as being homogeneous and isotropic. We can assimilate our observable universe to posses these qualities if the distance scale we consider is on the order of hundreds of mega-parsecs.
The evolution of the universe is given by Einstein equations together with the continuity equation for the substance that fill the universe. To solve the equations exactly we need the equation of state that gives the relation between pressure and energy-density of the substance.

Einstein equations

In General Relativity the evolution of the universe is given by the Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{1}$$

$G_{\mu\nu}$ is the Einstein tensor. In the right-hand side G is the universal gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor of the source field. $g_{\mu\nu}$ is the metric tensor and $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar, respectively.

Metric

The cosmological principle implies that the universe is homogeneous and isotropic. This can be given by the metric:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right). \qquad (2)$$

This is so because the metric has a spherical symmetry (homogeneous universe) and there are no crossed terms between time and space so no direction is privileged (isotropic universe).
k is the spatial curvature and the universe is:

- flat, if $k = 0$,
- closed (spherical), if $k = +1$,
- open (hyperbolic), if $k = -1$.

To derive the Firedman equations, first we need to calculate the Christoffel Symbols of the FLRW metric:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right). \tag{3}$$

Once the Cristoffel symbols have been calculated, we can calculate the Riemann tensor:

$$R^{\alpha}_{\beta\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\beta\mu}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\alpha}_{\beta\nu}}{\partial x^{\mu}} + \Gamma^{\lambda}_{\beta\mu}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\lambda}_{\beta\nu}\Gamma^{\alpha}_{\lambda\mu}. \tag{4}$$

The Ricci tensor is the contraction of the Riemann tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{5}$$

And the Ricci scalar:

$$R = R_{\mu\nu} g^{\mu\nu} = R^{\mu}_{\mu}. \tag{6}$$

Friedmann equations

Inserting the FLWR metric into the Einstein equations we obtain the Friedmann equations. The first is:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\varepsilon. \tag{7}$$

which is the 00 component of the Einstein equations. The second one is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3p). \tag{8}$$

Where:

- a is the scale factor,
- $H = \frac{\dot{a}}{a}$ is the Hubble parameter,
- G is Newton's gravitational constant,
- ε is the energy-density,
- p is the pressure.

Continuity equation and equation of state

The Friedmann equations can not be solved on their own, so we need to introduce additional conditions. The equations of continuity, which in this case is the Bianchi identity:

$$\dot{\varepsilon} = -3H(\varepsilon + p), \tag{9}$$

and the equation of state for the substance:

$$p = f(\varepsilon). \tag{10}$$

Equations (7),(8),(9) and (10) form a completely determinate system and can be solved analytically.

Bringing the equation to a nicer form

We substitute $H = \dot{a}/a$ in (7) to get the following system:

$$\dot{H} = -H^2 - \frac{4\pi G}{3}(\varepsilon + 3p), \tag{11a}$$

$$\dot{a} = Ha, \tag{11b}$$

$$\dot{\varepsilon} = -3H(\varepsilon + p), \tag{11c}$$

$$p = f(\varepsilon). \tag{11d}$$

From the continuity equation (11c) and the equation of state (11d) we can calculate the expression for the energy-density as a function of the scale factor a, that is $\varepsilon = f(a)$:

$$\frac{d\varepsilon}{\varepsilon + f(\varepsilon)} = -3\frac{da}{a}. \tag{12}$$

Friedmann equations as function of redshift

To be able to utilize measured data we introduce the concept of redshift, z as:

$$z + 1 = \frac{a_0}{a}. \tag{13}$$

Thus we can rewrite equations (11) as function of redshift instead of time:

$$\frac{dH}{dz} = \frac{1}{1+z}\left(H + \frac{4\pi G}{3H}(\varepsilon + 3p)\right) \tag{14a}$$

$$\frac{da}{dz} = -\frac{a_0}{(z+1)^2} \tag{14b}$$

$$\varepsilon = f(\frac{a_0}{z+1}) \tag{14c}$$

Equation of State

Let us consider the barotropic equation of state:

$$p = w\varepsilon,$$

where w is a constant known as EOS parameter. For usual a substance like perfect fluid w is positive, whereas for dark energy w is negative, precisely $w < -1/3$. Here we consider the cases when w takes the values $(0, 1/3, -1/2, -1)$. Which correspond to dust, radiation, quintessence, and cosmological constant.

Hubble parameter and scale factor as function of redshift

The differential equation for H:

$$\frac{1}{2}\frac{dH^2}{dz} = \frac{H^2}{z+1} + \frac{4\pi G\varepsilon_0}{3a_0^{3(1+w)}}(1+3w)(z+1)^{(2+3w)} \tag{15}$$

This equation allows the following analytical solution:

$$H(z) = \sqrt{(H_0^2 - \frac{8\pi G\varepsilon_0}{3})(z+1)^2 + \frac{8\pi G\varepsilon_0}{3}(z+1)^{3(1+w)}}, \tag{16a}$$

$$a(z) = \frac{a_0}{z+1}. \tag{16b}$$

Interpreting the Data

The data consists of the relative measured magnitude of type Ia supernovae and their measured redshift. In order to extract the value of the Hubble parameter from the data we have to numerically solve a system of differential equations to compute the distance to the supernovae.

$$\frac{dH}{dz} = \frac{H^2 + 4\pi G/3(\varepsilon + 3p)}{(z+1)H}, \tag{17a}$$

$$\frac{dD_T}{dz} = \frac{1}{(z+1)H}, \tag{17b}$$

where D_T is the light traveling distance.

Minimization

Having obtained the luminous distance we can now compute the theoretical values for the magnitude of the supernovae given by value for redshift:

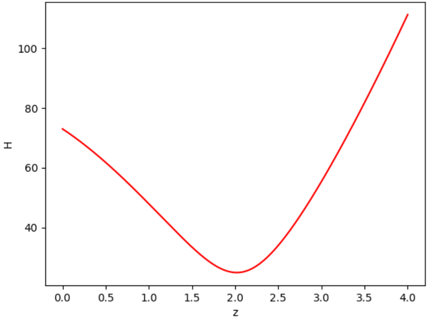
$$m = -m_0 \frac{5}{\log(100)} \log(\frac{1}{D_T^2}) + m_1 \tag{18}$$

In order to obtain the values for the energy-density and the cosmological constant at the present moment we have to minimize the sum:

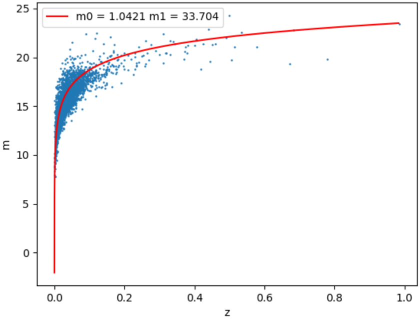
$$\sum (mag - mag_t(z))^2 \tag{19}$$

To do this we use the least-squares method and compute the best value for the initial energy-density, ε_0.

Dust filled Universe



(a) Hubble parameter as function of redshift



(b) Magnitude as function of redshift

Figure: W=0

Dust filled Universe

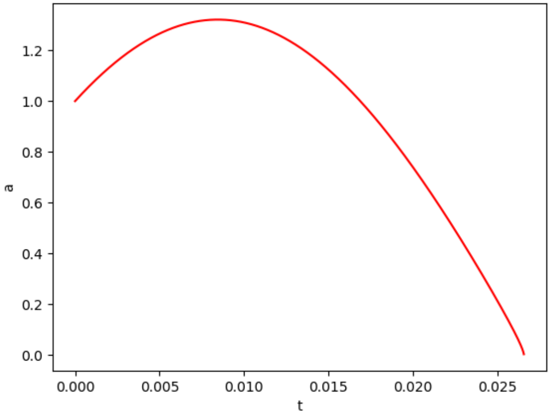
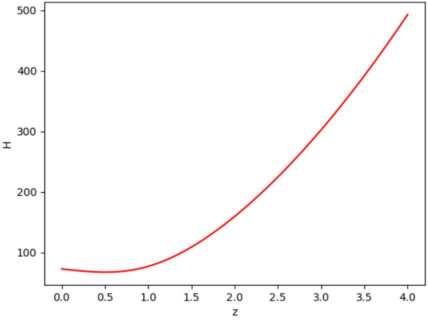
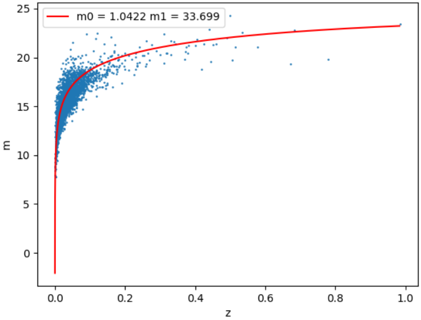


Figure: Scale factor as function of time, for the case W=0

Radiation filled Universe



(a) Hubble parameter as function of redshift



(b) Magnitude as function of redshift

Figure: W=1/3

Radiation filled Universe

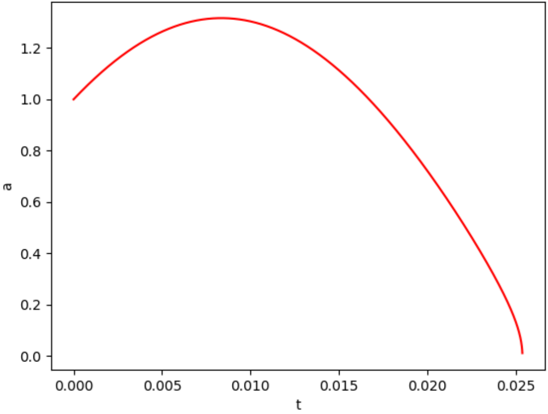
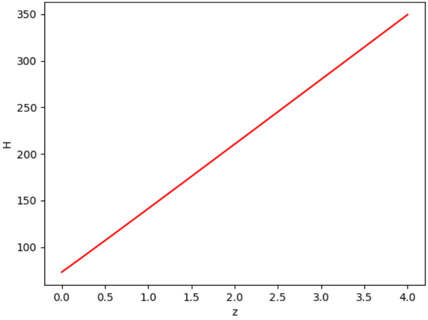
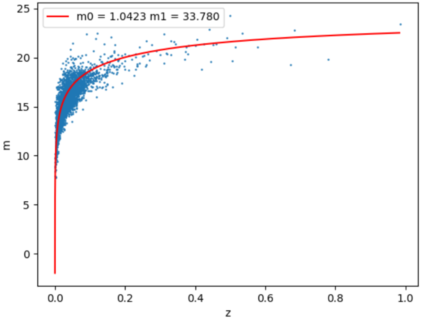


Figure: Scale factor as function of time, for the case W=1/3

Dark Energy filled Universe 1



(a) Hubble parameter as function of redshift



(b) Magnitude as function of redshift

Figure: W=-1

Dark Energy filled Universe 1

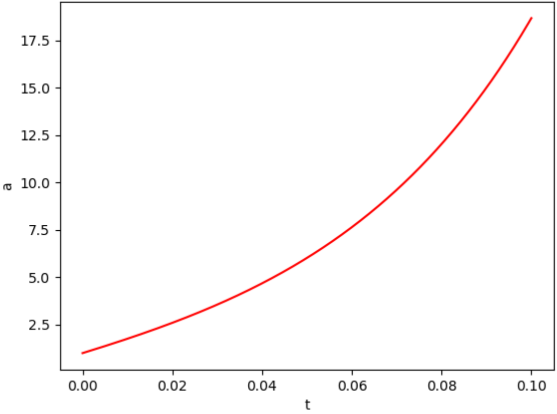
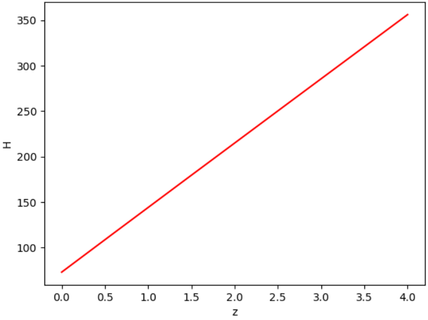
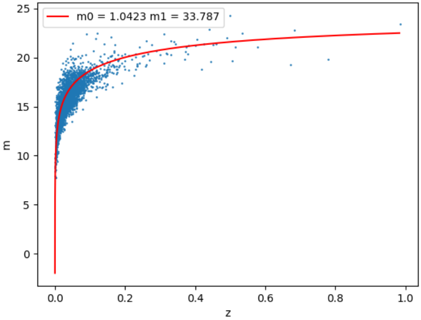


Figure: Scale factor as function of time, for the case W=-1

Dark Energy filled Universe 2



(a) Hubble parameter as function of redshift



(b) Magnitude as function of redshift

Figure: W=-1/2

Dark Energy filled Universe 2

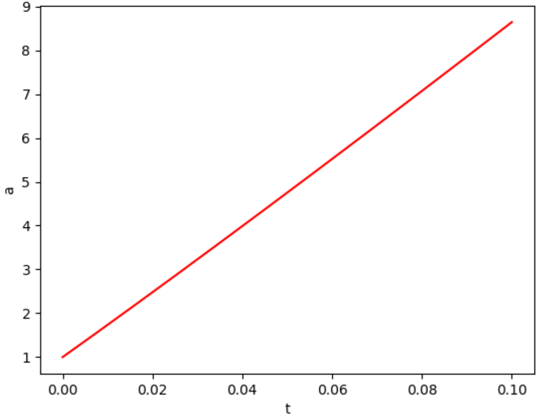


Figure: Scale factor as function of time, for the case W=-1/2

Sources

SAI Supernova Catalogue
http://stella.sai.msu.su/sncat/download.html
Viatcheslav Mukhanov
Physical Foundations of Cosmology